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ON THE GROWTH OF SMALL CAVITATION BUBBLES BY CONVECTIVE DIFFUSION

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Abstract—Some years ago beautifully conducted experiments on the growth of small cavitation bubbles were reported by Kermeen *et al.* An approximative calculation for the time in which bubbles grow to a certain size by convective diffusion was given by two of these authors as well. However, in this work vapor content of the bubbles and surface tension at the interface with the fluid were left out of account. In the present paper the theory is re-examined. Allowance is made for the above-mentioned effects. A model due to Levich is proposed for the calculation of the mass flux by diffusion in the bubbles. Results are obtained for the data occurring in the experiments of Parkin and Kermeen, and discussed in connection with these experiments. An explanation for the difference between theoretical and experimental values is suggested.

NOMENCLATURE

- c, concentration [kmoles/m^3];
- D, diffusion coefficient of air in water (2×10^{-9} m^2/s);
- k, gas constant ($8{\cdot}3 \times 10^3$ J/kmole deg);
- L, length of strip in Fig. 2;
- M, molecular weight;
- N, number of kilomoles air in a bubble;
- p, static pressure;
- Pe, Péclet number;
- R, bubble radius;
- S, bubble surface;
- t, time [s];
- T, temperature [°K];
- U, fluid velocity far from bubble;
- $\mathbf{v}$, local fluid velocity;
- V, volume of bubble;
- x, coordinate along strip in equation (11);
- y, coordinate normal to strip in equation (11);
- ρ, density;
- σ, coefficient of surface tension.

Subscripts

- g, gas;
- f, fluid;
- v, vapor;
- ∞, at infinity;
- 0, initial value in R_0, equilibrium value in c_0.

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1. INTRODUCTION

SOME years ago Kermeen *et al.* [1] and Parkin and Kermeen [2] published beautiful experiments on the growth of small cavitation bubbles in the boundary layer on a body immersed in a water stream of high velocity. The body consisted of a hemispherical head, smoothly connected with a circular cylinder. In the boundary-layer region downstream from the pressure minimum small bubbles (radius of order 10^{-5} m) were observed and photographed, while growing in a position of relative rest with respect to the body. The growth of such bubbles is either determined by diffusion of dissolved gas from the fluid into the bubble (gaseous cavitation) or of a vaporous character (vaporous cavitation). The latter type of growth bears an explosive character (see Section 2 below) and because in the experiments mentioned above such an explosive growth did not occur, it was concluded in reference [2] (henceforth denoted with P.K.) that in this case the bubbles were growing by diffusion of gas (mainly air) from the ambient water into the bubble. For bubbles at rest with respect to the fluid growth times by diffusion have been calculated by Epstein and Plesset [3]. In P.K. the growth times measured were found to be orders of magnitude smaller than those predicted by the theory of reference [3]. This discrepancy was in P.K. attributed to convective diffusion, which process does not occur in the situation dealt with in [3]. In [2] the authors developed an approximate theory for the growth by convective diffusion.

The growth times calculated with this theory were found to agree reasonably well with the times actually measured. The experimental data reproduced from the work of P.K. (reference [2]) and the graphs following from the theory in reference [2] (formula 13 of reference [2]) are shown in Fig. 1.*

In P.K. the effects of surface tension and vapor content on the bubble growth were not considered. It is shown in Section 2 below that in the relevant circumstances these pressures are of the same order of magnitude as the static pressure in the fluid. In the present paper an attempt is made to improve the theory in P.K. by taking the effects of vapor pressure and surface tension into account.

Also a model for the calculation of the mass flux in the bubble is used, which is thought to be an improvement with respect to the model used in P.K.

The object of the present paper is to investigate whether by these amendments the agreement with the measured growth times, reported in P.K., can be improved.

2. THE GROWTH OF SMALL BUBBLES BY AIR DIFFUSION

We consider small bubbles, filled with vapor and air, with radius R in a fluid with pressure p_f and velocity U relative to the bubble.† We denote the temperature of the fluid with T, the vapor pressure with p_v and the coefficient of surface tension at the fluid–gas interface with σ. In the experiments reported in P.K. $T = 294°$K, $p_v = 2{\cdot}28$ cm Hg $= 3100$ N/m^2, $\sigma = 7 \times 10^{-2}$ N/m. The size of the bubbles in P.K. is about 10^{-5} m, while $U \approx 8$ m/s, the values assumed by p_f are of the order of magnitude of p_v.

An important question is whether the bubbles assume a spherical shape under circumstances determined by the above mentioned values. Therefore we calculate the ratio between surface tension and pressure differences caused by inertia of the fluid

$$\frac{2\sigma}{R}/\tfrac{1}{2}\rho U^2.$$

This ratio, the Weber number, has a value of about 0·5. This means that the bubbles will not be exactly spherical but somewhat oblate. Because the Weber number is not very small we shall, however, deal with the bubbles as if spherical.

For a spherical bubble the growth is determined by the equation (see e.g. Plesset [4])

$$p_g + p_v - p_f - \frac{2\sigma}{R} = \rho\left\{R\frac{\mathrm{d}^2R}{\mathrm{d}t^2} + \frac{3}{2}\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2\right\}. \quad (1)$$

In (1) viscous forces are neglected; in addition to quantities already introduced

p_g = pressure of air in the bubble
ρ = density of water, equal to 10^3 kg/m^3
t = time.

The growth times reported in P.K. are of order 10^{-3} s. Then it follows that whereas the terms on the left-hand side of (1) are of order 10^3 N/m^2, those on the right-hand side are of order 10^{-1} N/m^2. Consequently we may omit the latter ones and consider the bubbles to be in equilibrium

$$p_f + \frac{2\sigma}{R} = p_g + p_v. \quad (2)$$

* We have found some difference between the formula (13) in [2] and the curves, derived from (13), of Fig. 3 in [2].

† The bubbles can be stationary with respect to the hemispherical head body because downstream from the pressure minimum the fluid friction is opposed by the adverse pressure gradient.

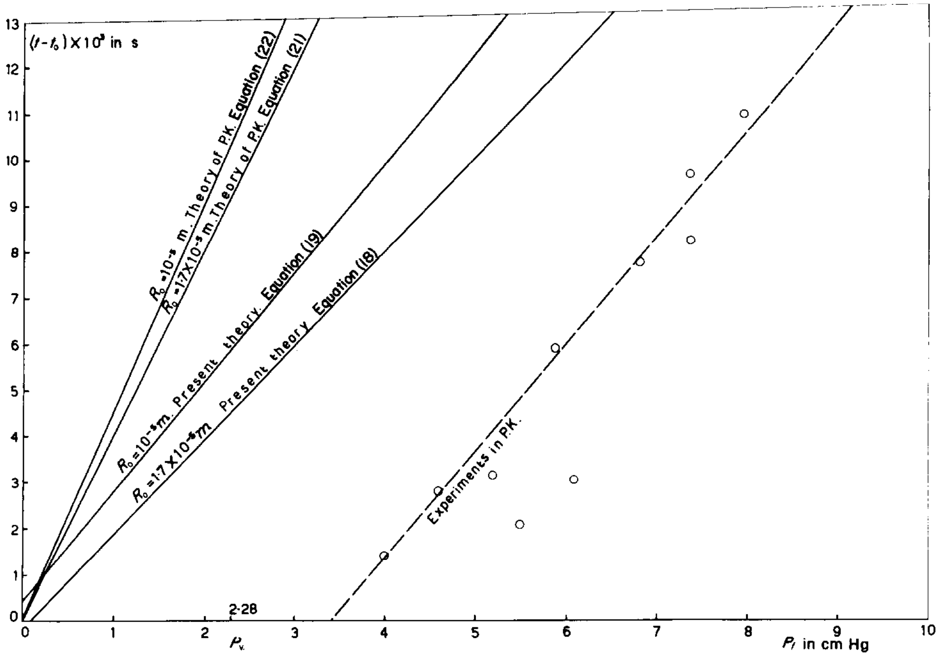

FIG. 1. Times for bubbles to grow from $R = R_0$ to $R = R_1 = 5 \times 10^{-5}$ m at various values of p_f.

The relationships (1) and (2) hold for a bubble at rest with respect to the fluid and p_f is the static pressure in the fluid. In our case, where there is relative motion, equation (2) indicates, if applied locally, that in regions of high p_f the curvature will decrease and in regions of low p_f increase, which results in an oblate shape. Averaged values over the bubble surface could then be used in (2) if the exact shape were known. Since this is not so we shall insert for p_f in (2) the static pressure in the absence of the bubble. To fix ideas we note that for a sphere the average static pressure is equal to the pressure far away minus $\frac{1}{4}\rho U^2$. For an ellipse the average static pressure equals the pressure far away at an axes ratio between 2 and 3.

Denoting the number of air kmoles in the bubble with N, the volume of the bubble with V and the universal gas constant with k, we write for the pressure p_g of air in the bubble

$$p_g = \frac{NkT}{V}. \tag{3}$$

Assuming that the growth is so slow that the process is isothermal, we obtain from (2) and (3)

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{kT}{p_f - p_v + (\frac{4}{3})(\sigma/R)} \frac{\mathrm{d}N}{\mathrm{d}t}. \tag{4}$$

If the denominator in the right-hand side of (4) vanishes, any nonzero $\mathrm{d}N/\mathrm{d}t$ causes an explosive growth. The value

$$p_f = p_v - \frac{4}{3}\frac{\sigma}{R} \tag{5}$$

marks the threshold for vaporous cavitation.

For the description of vaporous cavitation the terms in the right-hand side of (1) should be taken into account. Then a large, but finite rate of growth is obtained. The quantity $\mathrm{d}N/\mathrm{d}t$ is the kilomolar flux of air into the bubble. The driving agent for diffusion is the difference in air concentration far from the bubble, c_∞, and the equilibrium concentration c_0 at the bubble surface. The latter follows from the requirement that in equilibrium the thermodynamic potential must be minimum. For dilute solutions this leads to Henry's law (see e.g. Guggenheim [5]) stating that c_0 is proportional to p_g, the factor of proportionality being a function of temperature alone.

Because in P.K. c_∞ is in this sense associated with nearly atmospheric pressure and p_g is of order of 10^3 N/m^2, c_0 is negligibly small with respect to c_∞. Due to the difference between c_∞ and c_0, there is a concentration gradient in the fluid. The mass flux is connected with the component $\partial c/\partial n$ of this gradient normal to the bubble surface by

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \int D \frac{\partial c}{\partial n} \mathrm{d}S \tag{6}$$

where D is the diffusion coefficient for air in water with the value $D = 2 \times 10^{-9}$ m^2/s and dS is a surface element. In the following Section we occupy ourselves with the calculation of dN/dt.

3. CALCULATION OF THE RATE OF CONVECTIVE DIFFUSION

The equation governing the concentration distribution in the fluid is, **v** being the velocity vector in the fluid,

$$\frac{\partial c}{\partial t} + \mathbf{v} \,.\, \nabla c = D \,\nabla^2 c \tag{7}$$

with the boundary conditions

$$c = c_\infty \text{ at infinity} \tag{8}$$

$$c = c_0 \text{ at the bubble surface.} \tag{9}$$

Following P.K. we note that a representative velocity of the fluid is the velocity, U say, at displacement thickness of the boundary layer in which the bubble is located. The observed growth times being between 10^{-3} and 10^{-2} s, it follows that for $U \approx 10$ m/s and $R \approx 10^{-5}$ m, the ratio between the second and the first terms in the left-hand side of (7) is at least 10, so that we can regard the diffusion process as steady:

$$\mathbf{v} \,.\, \nabla c = D \,\nabla^2 c. \tag{10}$$

For solution of (8–10) the velocity distribution around the bubble in the boundary layer on the hemispherical head body must be known. The whole problem presents tremendous difficulties and in order to make progress, some approximations have to be made. We discuss first briefly the approximate calculation in P.K. reference [2]. There the bubble is represented by a two-dimensional strip of the rather arbitrary width $(\pi)^{\frac{1}{2}}R$. This strip is part of an otherwise impermeable wall (see Fig. 2) along which water flows with a homogeneous velocity U. If the x

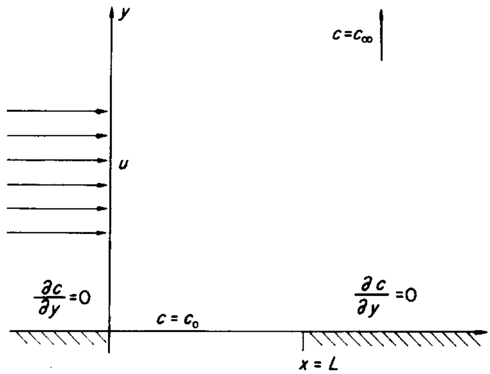

FIG. 2. Representation of approximated diffusion problem in P.K.

coordinate is in the direction of the strip and and the y coordinate normal to it, (10) reduces to

$$U\frac{\partial c}{\partial x} = \frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2}. \tag{11}$$

If the strip extends from $x = 0$ to $x = L$, the boundary conditions are

$$c = c_0 \quad \text{at} \quad y = 0; \quad 0 < x < L \tag{12}$$

$$\frac{\partial c}{\partial y} = 0 \quad \text{at} \quad y = 0; \quad x < 0, \quad x > L \tag{13}$$

$$c = c_\infty \quad \text{at} \quad (x^2 + y^2)^{\frac{1}{2}} \to \infty. \tag{14}$$

Also this problem cannot be solved exactly and further approximations have to be made. The method used in P.K. for the approximate solution of (11–14) is together with other approximative solutions of these equations discussed elsewhere by the present author [6].

Meanwhile one might look for a more realistic model to represent the mass transfer to the bubble. Here we suggest a model, which we think to offer some improvement and which is due to Levich [7]*. Levich argues that there is on the bubble no velocity boundary layer in the usual sense, because there is no condition for the tangential velocity except for the condition that at the gas–water interface the stresses are continuous. Evaluation of this idea leads to the conclusion that for sufficiently high Reynolds number (for the experiments in P.K. this is about 10^2, which is according to Levich sufficiently high) there is no region in the fluid where viscous forces dominate or equal inertia forces. To a high degree of accuracy therefore the velocity distribution around the bubble is given by the inviscid flow around the bubble.† Another important observation is that for this type of mass transfer the Péclet number, $Pe = UR/D$ is high. For the experiments in P.K. a representative value is $Pe = 10^5$. This means that diffusion takes place in a narrow region around the bubble, the diffusion boundary layer. The thickness of this boundary layer is (see for example reference [7], p. 407) of order $(RD/U)^{\frac{1}{2}}$, so that the ratio between boundary-layer thickness and bubble radius is given by $Pe^{-\frac{1}{2}}$. Under these circumstances the derivatives of c normal to the bubble are large with respect to those along the bubble. Omitting the latter in the right-hand side of (10) and inserting for $\mathbf{v}$ the velocity distribution at the bubble wall obtained from the inviscid flow round the bubble at oncoming velocity U, Levich obtains an equation for the concentration which is solvable in terms of error functions, yielding for the integral in (6)

$$\frac{\mathrm{d}N}{\mathrm{d}t} = 4(c_\infty - c_0)(2\pi R^3 UD)^{\frac{1}{2}}. \tag{15}$$

We note that for the present case the velocity distribution round the bubble, not based on a homogeneous primary flow, but on the velocity profile in the boundary layer on the hemispherical head body, should be used. We have not attempted to determine this velocity distribution on the bubble. Work in that direction has been done by Lighthill [8, 9], for weak velocity gradients.

Here we have restricted ourselves to a homogeneous primary flow, with a representative velocity U.

The relation corresponding to (15) in the approximate theory in P.K., with the width and breadth of the strip taken as $L = R\sqrt{\pi}$, is

$$\frac{\mathrm{d}N}{\mathrm{d}t} = 4(c_\infty - c_0)\,\pi^{\frac{1}{4}}\,(R^3 UD)^{\frac{1}{2}}. \tag{16}$$

The value given by (15) is about twice higher.

4. CALCULATION OF GROWTH TIMES

The growth by diffusion of a bubble follows from substitution of (15) in the right-hand side of the relation (4). We obtain that for growth from radius R_0 to radius R_1 the necessary time is given by

$$t_1 - t_0 = \frac{\pi^{\frac{1}{2}}}{(2UD)^{\frac{1}{2}}\,kTc_\infty} \times \int_{R_0}^{R_1} \left(p_f - p_v + \tfrac{4}{3}\frac{\sigma}{R}\right) R^{\frac{1}{2}}\,\mathrm{d}R. \tag{17}$$

Growth times according to (17) were calculated pertaining to the values of the various parameters reported in P.K. These values are listed below.

R_0 is the initial radius of a bubble, R_1 the final radius. Following P.K. we calculated the growth time $t_1 - t_0$ as a function of pressure for two values of R_0. Because the effect of

* The author is indebted to Dr. Marshall P. Tulin for bringing this book to his attention.

† We note that the effect of surface active contaminants may alter this, particularly (see, for example, reference [7], Chapter 8) for bubbles at small Reynolds number.

photographic resolution on the smallest observable size was not exactly known, these two values were chosen in P.K.

In P.K. the calculations were carried out for two velocities, the velocity at the edge of the velocity boundary layer at the hemispherical head body (50 ft/s) and the velocity at the displacement thickness (28 ft/s). We carried out the calculations for a velocity of 8·4 m/s which is about equal to the latter velocity.

The data used (taken from P.K., reference [2]) are

$$R_0 = \begin{matrix} 1{\cdot}7 \times 10^{-5}\,\mathrm{m} \\ 1 \times 10^{-5}\,\mathrm{m} \end{matrix} \quad R_1 = 5 \times 10^{-5}\,\mathrm{m}$$

$$U = 8{\cdot}4\,\mathrm{m/s};\ p_v = 2{\cdot}28\,\mathrm{cm\,Hg} = 3100\,\mathrm{N/m^2}$$

$$\sigma = 7 \times 10^{-2}\,\mathrm{N/m}$$

$$D = 2 \times 10^{-9}\,\mathrm{m^2/s}$$

$$c_\infty = 5 \times 10^{-4}\,\mathrm{kmoles/m^3}$$

$$k = 8{\cdot}3 \times 10^3\,\mathrm{J/kmole\ deg}$$

$$T = 70°\mathrm{F} = 294°\mathrm{K}.$$

The results are:

for $R_0 = 1{\cdot}7 \times 10^{-5}$ m

$$t_1 - t_0 = 4{\cdot}5 \times 10^{-3} + 2{\cdot}1 \times 10^{-3} \times (p_f - p_v), \tag{18}$$

for $R_0 = 10^{-5}$ m

$$t_1 - t_0 = 5{\cdot}9 \times 10^{-3} + 2{\cdot}4 \times 10^{-3} \times (p_f - p_v). \tag{19}$$

In the above equations p_f and p_v are in cm Hg and $t_1 - t_0$ in seconds. The lines representing the relationships (18) and (19) are drawn in Fig. 1.

In the theory in P.K. the effects of surface tension and vapor content are left out of account and the mass transfer is calculated according to (16). In that case the relation (17) is

$$t_1 - t_0 = \frac{\pi^{\frac{1}{2}} p_f}{(UD)^{\frac{1}{2}} kTc_\infty} \int_{R_0}^{R_1} R^{\frac{1}{2}}\,\mathrm{d}R. \tag{20}$$

Since the equilibrium relation (2) simply is

$$p_f = p_g$$

the ratio p_f/kT is apart from a numerical constant equal to the density ρ_g of the air in the bubble, whence it follows that

$$t_1 - t_0 = \frac{\pi^{\frac{1}{2}} \rho_g}{(UD)^{\frac{1}{2}} c_\infty} \int_{R_0}^{R_1} R^{\frac{1}{2}}\,\mathrm{d}R$$

This relation is identical with the relation (13) in reference [2]. Calculating the growth times according to (20) we obtain

$$t_1 - t_0 = 4 \times 10^{-3} p_f; \quad R_0 = 1{\cdot}7 \times 10^{-5}\,\mathrm{m} \tag{21}$$

$$t_1 - t_0 = 4{\cdot}55 \times 10^{-3} p_f; \quad R_0 = 10^{-5}\,\mathrm{m} \tag{22}$$

As already mentioned in the introduction, we have found some difference between (21) and (22) on one hand and the curves shown in Fig. 3 of reference [2] on the other hand.

5. DISCUSSION

Although in the order of magnitude analysis in P.K. surface tension and vapor content are not taken into account it follows from comparison of the lines in Fig. 1 that for not too large values of $p_f - p_v$ the differences with the results of the present theory are rather small numerically. The reason for this is that with the data used for the calculation the vapor pressure is of the same magnitude as surface tension, so that the term between brackets in (17) does not differ much from p_f. The main numerical difference is caused by the factor $(2\pi^{\frac{1}{2}})^{\frac{1}{2}}$ by which the mass flux is larger in the present theory.

The dashed line through the experimental points is represented by

$$t_1 - t_0 = -2{\cdot}6 \times 10^{-3} + 2{\cdot}3 \times 10^{-3} (p_f - p_v). \tag{23}$$

The slope of this line is between those obtained with the present theory for different values of R_0.

From consideration of Fig. 1 and from comparison of (23) with (18) and (19) it follows that agreement between the present theory and the experiments in P.K. could be obtained by a shift of the dashed line to the left over a distance corresponding with 3·40 cm Hg. For such a shift the following explanation is suggested here. Extrapolating the experimental data, we find that the dashed line in Fig. 1 intersects the p_f-axis at a value larger than p_v. This would suggest that vaporous cavitation ($t - t_0 = 0$ for growth to any size) starts at a pressure in the fluid larger than p_v, which is impossible. However, for p_f the local pressure in the absence of bubbles is inserted. In Section 2 we discussed the effect of the bubble shape on the average static pressure. For a sphere the average pressure is $p_f - 0{\cdot}25\rho U^2$. For the oblate shape which the bubble assumes under the influence of surface tension and pressure gradient the difference between p_f and the average pressure will be smaller. With a velocity of 8·4 m/s, $\rho U^2 = 7 \times 10^4$ N/m^2. A shift of the dashed line in Fig. 1 by an amount of 0·065 of this value to the left would result in coincidence of the dashed line obtained from the experiments with a line in between those following from the present theory (equations 18 and 19). We have not attempted to calculate the average pressure on the bubble, since it would be very difficult indeed to obtain an accurate enough description of the bubble shape under the conditions of the experiments in P.K.

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Résumé—Il y a quelques années de belles expériences sur la croissance de petites bulles de cavitation ont été décrites dans un article de Kermeen et al. Deux des auteurs de cet article ont donné également un calcul approché du temps nécessaire pour que les bulles atteignent une certaine taille par diffusion convective, mais en laissant de côté la teneur en vapeur des bulles et la tension superficielle à l'interface fluide. La théorie est reéxaminée ici, en tenant compte des effets mentionnés ci-dessus. Un modèle dû à Levich est proposé pour le calcul du flux massique pour diffusion dans les bulles. Des résultats sont obtenus pour les données des expériences de Parkin et Kermeen, et discutés en les comparant avec ces expériences. On suggère une explication pour la différence entre les valeurs théoriques et expérimentales.

Zusammenfassung—Vor einigen Jahren wurde über gut durchgeführte Versuche über das Wachstum von Blasen an kleinen Hohlräumen von Kermeen und anderen berichtet. Dabei wurde eine Näherungsrechnung für die Zeit in der die Blasen infolge konvektiver Diffusion zu einer bestimmten Grösse anwachsen von zwei der Autoren angegeben. Jedoch wurde in dieser Arbeit der Dampfgehalt der Blasen und die Oberflächenspannung an der Grenzfläche zur Flüssigkeit nicht in die Betrachtung einbezogen. In der vorliegenden Arbeit wird die Theorie überprüft und die oben erwähnten Einflüsse werden berücksichtigt. Für die Berechnung des Massenstromes durch Diffusion in die Blasen wird ein Modell nach Levich vorgeschlagen. Für Messwerte aus den Ergebnissen von Parkin und Kermeen wurden Ergebnisse erhalten und im Zusammenhang mit diesen Versuchen diskutiert. Eine Erklärung für den Unterschied zwischen theoretischen und experimentellen Werten wird vorgeschlagen.

Аннотация—Несколько лет тому назад Кермееном и другими были опубликованы прекрасно проведенные эксперименты по росту малых кавитационных пузырьков. Они же дали приближенный расчет времени роста пузырьков до определенного размера благодаря конвективной диффузии. Все же в этой работе остались неучтенными паросодержание пузырьков и поверхностное натяжение на границе раздела с жидкостью.

Теоретические расчеты в настоящей работе рассматриваются заново. Принимаются во внимание упомянутые выше эффекты. Для расчета диффузионного массового потока в пузырьках предлагается модель Левича. Полученные результаты, в общем, удовлетворительно согласуются с экспериментальными данными Паркина и Кермеена. Предлагается объяснение расхождения между теоретическими и экспериментальными значениями.